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INTRODUCTION TO MULTIPLE STATE MULTIPLE ACTION DECISION THEORY AND ITS RELATION TO MIXING STRUCTURES

ARMY ELECTRONICS COMMAND
FORT MONMOUTH, NEW JERSEY

JANUARY 1977

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RESEARCH AND DEVELOPMENT TECHNICAL REPORT

ECOM-5810

INTRODUCTION TO MULTIPLE STATE MULTIPLE ACTION DECISION THEORY AND ITS RELATION TO MIXING STRUCTURES

By

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January 1977

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1. REPORT NUMBER ECOM-5810	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) INTRODUCTION TO MULTIPLE STATE MULTIPLE ACTION DECISION THEORY AND ITS RELATION TO MIXING STRUCTURES		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Bernard Francis Engebos		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Atmospheric Sciences Laboratory White Sands Missile Range, New Mexico 88002		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS DA Task 1L161102B53A
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Electronics Command Fort Monmouth, New Jersey 07703		12. REPORT DATE January 1977 13. NUMBER OF PAGES 59
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Decision theory
Game theory
Cost functions
Life cycle costs
Sensor mix
Figure of merit
Cost effectiveness
Mixing structure

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A general mathematical framework is developed which addresses the problem of determining an optimal, or near optimal, course of action, when the outcome of a given course of action is known to be influenced by an evolving state of nature. In this context the advantage of knowledge of the natural state is balanced by the cost of obtaining this information. Such a structure, when considered as functioning over a given time interval, permits employment of life cycle cost versus possible gain. All the mathematical structures and

20. Abstract (cont)

related entities, and the underlying properties thereof, are developed in a manner that such tradeoff studies are possible.

The theoretical development as presented is related to that of statistical game theory but with a broader set of objectives. Multiple aspects for the state of nature, and sets of permissible action are allowed, with these actions being capable of simultaneous performance. This leads to the introduction of multiple state multiple action decision theory and its basic framework, the "mixing structure."

The concept of "sensor mixes" is defined and related to the possibility of decreasing loss by the spying on the state of nature. The cost of obtaining this information is then balanced against the gain obtained by knowledge of the natural state. A resulting "figure of merit" may be used to determine the desirability of each sensor mix.

Several classes of decision functions are defined which demonstrate the properties a decision function must possess to be considered as desirable. An illustrative example for the structures defined in this thesis is then presented.

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CHAPTER I

INTRODUCTION

The basis for this thesis is the establishment of a mathematical framework of sufficient generality to perform cost versus operational effectiveness analyses for a class of physical problems. All involve the obtaining of information concerning an evolving "state of nature". It is desired in these problems to determine an optimal, or at least near optimal course of action, when the outcome of any permissible course of action is known to be influenced by the "state of nature". In this context, the advantages of knowledge of the natural state can be weighed against the "cost" associated with its acquisition.

This particular type of problem occurs frequently in both industrial and military applications. For instance, in the design of a system, it is desirable to assess the relative merits of the different possible configurations in terms of potential gain versus anticipated costs; i.e., to build a cost effective system. In developing a meteorological support system for a military application, it is possible to employ a wide range of meteorological sensors with varying degrees of accuracies and costs. A commander who has knowledge of this meteorological information may obtain an advantage over his adversary by more accurately employing his weapons and by more strategically positioning his men. It is desirable in this case to determine that combination of sensors which will yield maximal advantage for minimal expenditure.

For nonmilitary applications both cost and effectiveness are

usually expressed in a monetary standard. However in military applications, other criteria such as loss of manpower, material, etc., may also be used.

In many practical applications both the simultaneous performance of a set of actions and an evolving state of nature must be considered. Neither of these characteristics was found to be of sufficient generality in the literature to examine the problem of interest. Thus it was deemed necessary to consider a new approach.

The general flow of the presentation will be seen to be reminiscent of Blackwell and Girschick's classical work [2] on statistical games, but altered considerably to meet the criteria established in the previous paragraphs.

Other theories exist which are related to this general problem. Statistical team decision theory [12] is the study in which "several persons perform various tasks, including those of gathering and communicating information, and making decisions; but they have common interests and beliefs." This theory, including the works of Radner [16] and Rudge [17], does not permit an evolving state of nature except under very restrictive conditions. It also assumes that the alternative states of the environment are finite with known probabilities. In this thesis, no assumptions are made regarding the probabilistic aspects of the state of nature. Team theory also addresses the consequences of simultaneous performance of actions, but, in general, only under static conditions.

Several authors have examined problems relating to the value of

information. For example, Miller [13] considered the value of sequential information in a manner reminiscent of statistical game theory. Blau [3] performed a similar study involving nonzero sum stochastic games. Ho, Basar and Hexner [7] attacked this problem from the standpoint of stochastic optimization. In none of these cases did the authors coalesce an evolving state of nature with simultaneous performance of sets of actions.

There also exist additional areas ([1] and [15]) which offer possible avenues of approach to the problems presented herein. Most depend on a particular form of application.

In any case, each of the above-mentioned references has its specific area of application, but none has a sufficiently general structure as would be required to examine problems as described previously.

As will be seen in Chapter II, the state of nature is represented by a collection of real-valued functions, each representing a physical observable persisting over a duration T. Associated with each observable is a set of stochastic processes which represents the outputs of various "sensors" of that observable, each differentiated by its probabilistic nature.

At each time instant in T, a set A_t of possible actions is postulated, along with a class of subsets, $\hat{A}_t$, representing those groups of actions permissible at t. The actions in each of the various permissible sets at t have the context of "simultaneous performance".

If a permissible plan of action is initiated at t, the result

will be a realization of a random variable whose nature is dependent on the prevailing state of nature. The numerical "loss" will be the sole indicator of the advantage or disadvantage of a given plan of action. Structurally, a "loss" function is a bivariate set function whose domain is the Cartesian product of natural states and permissible plans of action, and whose range is a set of random variables of finite mean and variance defined on an appropriate probability space. This is the single most influential mathematical entity to be considered. In many applied problems the loss function is implicit but still must maintain these properties.

Any set function which transforms a realization of a set of sensors, or a "sensor mix" into an admissible plan of action, will be called a Multiple State, Multiple Action (MSMA) decision function. This function affords a link between the natural state and "loss" engendered by way of the earlier mentioned "loss", or MSMA loss function as it is called in the text.

Certain problems, among them the "sensor mix" problem are introduced, and possible approaches to solutions are offered. To accomplish this, a basic structure that is called a "Mixing Structure" is defined. This constitutes a fundamental structure which places all the above-mentioned entities in context.

Of interest, and reminiscent of statistical games, is the possibility of decreasing loss by "spying" on the state of nature, or some desired substate via a "sensor mix". The "cost" of such an action is balanced with the "gain" possible through this knowledge and a resulting "figure of merit" may be used to determine the

desirability of each sensor mix.

Since "loss" is itself subject to random influence, the "consistency of gain" becomes important and is examined. It is found that, in general, only decision functions with precisely defined properties are acceptable in context, and in this case, as accuracy increases, the loss approaches that which might be expected when the appropriate subset of the natural state is known. This is essential if cost and gain are to be balanced for a given sensor mix.

Chapter III presents an illustrative example of the structures previously defined and serves to demonstrate both existence and structure by providing a simple but comprehensive mixing structure. To highlight this example further, a simplified and somewhat naive version of an artillery problem is offered.

In Chapter IV certain open problems, either contemplated or under investigation, are presented.

It is realized that the bulk of this dissertation is given to the establishment and verification of a mathematical structure of sufficient generality to consider a specific family of applied problems. The complexity of the problems which are considered are of such a scope that a fair volume of material is left undone; however, the interest and importance of answering these problems are felt to justify this development. This ultimately should prove beneficial to several fields of endeavor.

CHAPTER II

BASIC STRUCTURES

Let $\{f_p \mid p \in P\}$ be an indexed set of real-valued functions defined and bounded on a real interval T. This family will represent distinct physical observables persisting throughout a specified interval T of time. For a fixed $t \in T$, the real number $f_p(t)$ will be called the state of f_p at time t. The set $\{f_p(t) \mid t \in T\}$ will be called the evolution of f_p over T. The fact that the observables in $\{f_p \mid p \in P\}$ are distinct assumes that no two members of this family have the same evolution. Sets of the form $\{f_p \mid p \in P\}$ will be of special interest. For each $t \in T$, the set

$$S_t = \{f_p(t) \mid p \in P\}$$

will be called the state of nature at t.

Associated with each $t \in T$ will be a nonempty set A_t called the set of actions at t. For each such set of actions, a class of subsets $\hat{A}_t$ will be assumed where any B_t in $\hat{A}_t$ is called an admissible plan of action at t. Actions in B_t will have the connotation of simultaneous performance; hence, if an action "a" is found in B_t, its negative counterpart is excluded. The indexed set $\{(A_t, \hat{A}_t) \mid t \in T\}$ will be called the family of actions over T. Any family $\{B_t \mid t \in T, B_t \subset \hat{A}_t\}$ will be called an admissible course of action. It will be assumed that sets in $\hat{A}_t$ possess the hereditary property for any t in T.

In the structure to follow, distinction will be made between instantaneous properties specific to some fixed t in T, and pro-

perties relating to the entire interval T. As an example, the state of nature at t is an instantaneous characteristic, while the evolution of f_p relates to the entire interval.

Basic to the structure is the notion that any admissible plan of action performed at time t will result in an outcome which is to a greater or lesser extent influenced by the prevailing state of nature, and by random external influences. An underlying concept which will serve to bind together the various component structures is that instantaneously, the outcome of initiating an admissible plan of action may be characterized by a single numerical value of "loss". Moreover, the possibility exists of diminishing loss through the choice of a plan of action based on "a priori" knowledge concerning the state of nature. Also, the possibility will be considered that knowledge of the state of nature may be neither complete nor exact. In fact, it may be subject to errors of a random nature.

In effect, the possibility will be considered of "spying" on the state of nature with the purpose in mind of minimizing loss. To place the notion of "spying" on a more precise and tractable ground, the following structure is introduced.

Let (R,L,m) be the measure space consisting of the real numbers R, the class L of Lebesque measurable subsets of R, and the Lebesque measure m. Let Ω be the set of all stochastic processes on this space which are indexed by T and which have the further property; if $\{\omega_t \mid t \in T\}$ is in Ω, there exists a state variable f_p whose evolution is exactly $\{E(\omega_t) \mid t \in T\}$, where E denotes the expected value operator. The set Ω may be visualized as containing

all unbiased sensing mechanisms for the physical observables in $\{f_p \mid p \in P\}$. It will be through the members of Ω that the ability to "spy" on the state of nature will be manifested.

Certain natural maps may be defined on Ω which prove useful from a notational or expository standpoint. First, for $s \in T$, let $g_s^* : \Omega \to R$ be defined by the relation

$$g_s^*[\{\omega_t \mid t \in T\}] = E(\omega_s)$$

where $\{\omega_t \mid t \in T\} \in \Omega$. It may be easily verified that g_s^* is a map as specified. Moreover, for each $s \in T$,

$$g_s^*(\Omega) = S_s,$$

where S_s is the state of nature at s.

A second useful relation carries Ω into $\{f_p \mid p \in P\}$. It is defined by letting the image of $\{\omega_t \mid t \in T\}$ in Ω be that state variable f_p whose evolution is $\{E(\omega_t) \mid t \in T\}$. The fact that the functions $\{f_p(t) \mid p \in P\}$ are distinct dictates that the relation is indeed a map. As such, it will be given the designation h^*.

A final map which will prove useful may be defined by letting the image of $\{\omega_t \mid t \in T\}$ be a composite map of h^* and the index function of $\{f_p \mid p \in P\}$; that is, if the function is designated j^*, $j^*[\{\omega_t \mid t \in T\}]$ will be the index for the image of $\{\omega_t \mid t \in T\}$ under h^*.

It is worthwhile to note that in a straightforward manner, the function h^* may be seen to determine a partition of Ω; i.e., $\bigcup_{p\in P} (h^*)^{-1}(f_p) = \Omega$ and $(h^*)^{-1}(f_p) \cap (h^*)^{-1}(f_q) = \phi$ if $p \neq q$. This partition will be called the <u>h^*-partition</u> of Ω. For each $p \in P$,

$(h^*)^{-1}(f_p)$ will be called the <u>set of sensors</u> of f_p. It now becomes possible to define one of the basic mathematical entities which figures in the following development.

DEFINITION 2.1. Let $\{f_p \mid p \in P\}$, Ω, and all those structures implicit in their construction be as previously defined. A <u>sensor mix in Ω</u> will be defined as any finite subset of Ω which meets each member of the h^*-partition at no more than a single point. The set of all sensor mixes in Ω will be denoted by Γ.

In most practical applications the cardinality of P will be finite, although for what follows, the cardinality of P will be considered to be at most countable. The assumption that each sensor mix contains no more than a single element of any member of the h^*-partition is not as restrictive as it may at first seem. Concern will be with finite sensor sets, hence within the general scope and interest of the investigations to follow, any finite sensor set belonging to a single physical observable may be replaced by a single sensor which can provide probabilistically equivalent information.

In an unambiguous fashion, h^*, g_s^*, and j^* define maps from Γ into $\{f_p \mid p \in P\}$, R, and P respectively, where s is in T. From this point on these maps will be denoted by h, g_s, and j respectively.

THEOREM 2.2. If Γ is the class of sensor mixes as previously defined, the following behavior traits are exhibited:

a) For any $Q \subset P$, if $\{\gamma_q \mid q \in Q\}$ is a family of sensor mixes, then $\bigcap_{q \in Q} \gamma_q$ is a sensor mix.

b) If $\mu,\gamma \in \Gamma$, and $h(\mu) \cap h(\gamma) = \phi$, then $\mu \cup \gamma \in \Gamma$.

c) If $\mu,\gamma \in \Gamma$ and $h(\mu) \cap h(\gamma) \neq \phi$, then $\mu \cup \gamma \in \Gamma$ if and only if $f_p \in h(\mu) \cap h(\gamma)$ implies that $h^{-1}(f_p) \cap \mu = h^{-1}(f_p) \cap \gamma$.

d) If $\mu,\gamma \in \Gamma$, then $\mu - \gamma$ and $\gamma - \mu$ are in Γ. Thus the symmetric difference $\gamma \circ \mu$ is in Γ.

Proof: The proof is elementary and will be omitted.

Part c) of this theorem states that if two sensor mixes are such that their images under h contain a common physical observable, then the two sensor mixes must coincide at that physical observable if their union is to be a sensor mix.

A sensor mix as previously defined will serve as the basic unit of estimation whenever "a priori" information is desired concerning the state of nature.

At this point the problem of a suitable space for the realizations of the various members of Γ arises. Toward this end, it may be noted that any realization of a nonempty sensor mix will be of the form $\{\alpha_q \mid q \in Q \subset P\}$, where for each $q \in Q$, α_q is a real number and Q has finite cardinality. Observe that without ambiguity such a vector may always be identified with a function of Q into R. In fact, the set of realizations for a sensor mix γ may always be identified with the set of real-valued functions on $j(\gamma)$. This fact is a motivation for the following.

Let $\hat{Q}$ be the class of all finite subsets of P and $Q \in \hat{Q}$. Define $X_Q = \{x \mid x : Q \to R\}$ and let $X = \bigcup_{Q \in \hat{Q}} X_Q$. The convention

will be made that $\phi \varepsilon \hat{Q}$ where ϕ will be called the <u>empty sensor mix</u>. Then X_ϕ will be defined to be ϕ, and will by abuse of the language be considered the "realization" of the empty sensor mix.

It is now clear that any realization of a sensor mix γ may be identified with a unique member of $X_{j(\gamma)}$. In the material to follow, this identification will be assumed, and again by the abuse of the language, a realization of γ will be spoken of as lying in $X_{j(\gamma)}$.

Adequate structure has been introduced to permit the definition of perhaps the single most influential mathematical entity in the study to follow, the <u>multiple state multiple action (MSMA) loss function</u>. This function will serve to establish the link between an observation of some subset of the state of nature and the outcome of any course of action which might be taken.

DEFINITION 2.3. Let Λ be the set of all random variables on (R,L,m) of finite mean and variance. A function L_t which carries $X \times \hat{A}_t$ into Λ will be called an <u>MSMA loss function</u> at t. A family $\{L_t \mid t \varepsilon T\}$ will be called a <u>progression of losses</u>.

DEFINITION 2.4. Let $\{f_p \mid p \varepsilon P\}$, $\{(A_t,\hat{A}_t) \mid t \varepsilon T\}$, Ω, X, $\{L_t \mid t \varepsilon T\}$, and Λ, along with all structures inherent in their composition be as previously defined. This sextuple of mathematical entities will be designated as a <u>mixing structure over T</u>. It will be denoted by M.

In the next chapter it will be demonstrated that nontrivial mixing structures exist and a specific mixing structure will be exhibited. At present, existence will be assumed.

If a realization of a sensor mix is known, any function which

links this realization with a plan of action will play the role of an MSMA decision function.

DEFINITION 2.5. Let M be a mixing structure over T. Let δ_t be any function carrying X into $\hat{A}_t$. Such a function will be called an MSMA decision function at t. A family $\{\delta_t \mid t \in T\}$ of MSMA decision functions will be called a decision scheme for M.

Note that since $\phi \in X$, $\delta_t(\phi)$ must represent a plan of action at ϕ. This plan of action will correspond to a decision based on an absence of any knowledge of the state of nature. This plan of action $\delta_t(\phi)$ will be called the standard plan of action for the MSMA decision function δ_t.

With this definition of an MSMA decision function, the instantaneous operation of a mixing structure will now be defined. One judiciously chooses some sensor mix $\gamma \in \Gamma$ and an MSMA decision function δ_t. At time $t \in T$, the sensor mix γ provides an estimate of the state of nature. The decision function δ_t determines, based on a realization of γ, a plan of action which is then initiated. This plan of action permits an assessment of advantage through a realization of the appropriate member of Λ. This numerical indicator of loss determines the advantage or disadvantage of the operation and completes an operational cycle.

Observe that no loss of time is assumed to occur between the determination of a realization of the chosen sensor mix and the initiation, completion, and appraisal of the appropriate plan of action. If such a condition holds throughout T, the mixing structure is said to be a mixing structure without lag. For the

present, interest will be with such structures.

In an examination of mixing structure operation, it is seen that primary importance is placed on pairs of the form $(\gamma, \{\delta_t \mid t \in T\})$; i.e., pairs consisting of a sensor mix and a decision scheme. This leads to the following problems.

PROBLEM I: If a decision scheme is specified, the problem of determining γ (optimal γ) when a "cost" consideration is given will be called a <u>sensor mix</u> (<u>optimal sensor mix</u>) <u>problem</u>.

PROBLEM II: If γ is given, the problem of determining $\{\delta_t \mid t \in T\}$ will be called an <u>MSMA decision problem</u>.

PROBLEM III: Let $\Gamma' \subset \Gamma$ be given, and for each $\gamma' \in \Gamma'$, let c be defined such that c is a real-valued, nonnegative set function defined on Γ' having the property that for $\mu, \nu \in \Gamma'$ and $\mu \subset \nu$, then $c(\mu) \leq c(\nu)$. c will be called a cost function for Γ'. If (Γ', c) is given for some mixing structure, the determination of a rationale for choosing a pair $(\gamma, \{\delta_t \mid t \in T\})$ will be called a <u>fundamental sensor mix problem</u>. If constraints are included, the problem will be called a <u>fundamental sensor mix problem with constraints</u>.

The remainder of this thesis is devoted to the development of a convenient mathematical structure for the exploration of such problems.

Pertinent to the problems mentioned above, and to the general theory of sensor mixes, is a special class of functions in X. This set of functions is related to the set of states of nature at a given time $t \in T$, or in other words, to $\{S_t \mid t \in T\}$. Define

$$s_t : P \to R : p \to f_p(t).$$

It is clear that for any finite $Q \subset P$, $s_{t|Q} \in X_Q$; i.e., s_t restricted to a finite set Q is a member of $X_Q \subset X$.

Concern will be with various finite subsets of S_t. By a slight abuse of the language, an identification will be made between a subset U of S_t and the function $s_{t|Q}$ in X where $U = \{f_q(t) \mid q \in Q \subset P\}$. Thus $L_t[s_{t|Q}, \delta_t(s_{t|Q})]$ will be written as $L_t(U, \delta_t(U))$. No confusion should result from this identification.

One basic characteristic of a MSMA decision function may now be defined.

DEFINITION 2.6. An MSMA decision function δ_t is said to be <u>consistent</u> <u>with</u> <u>the</u> <u>state</u> <u>of</u> <u>nature</u> if and only if for $U, V \subset S_t$ such that $U \subset V$ it follows that

a) $E[L_t(V, \delta_t(V))] \leq E[L_t(U, \delta_t(U))]$

b) $E[L_t(V, \delta_t(V))] \leq E[L_t(V, \delta_t(U))]$

c) $E[L_t(U, \delta_t(\phi))] = E[L_t(V, \delta_t(\phi))]$.

The motivation behind Definition 2.6 has an intuitive interpretation. Condition a) states that if an MSMA decision function δ_t is consistent with the state of nature, decisions based on an increasing knowledge of the state of nature will not result in an increase of expected loss. Condition b) reflects the property that for any given state of nature, one cannot decrease expected loss by basing a decision on some smaller subset. Finally, condition c) is a statement that whenever a decision is made to use the standard plan of action, expected loss is independent of the state of nature observed.

As will be seen, an MSMA decision function consistent with the state of nature has many desirable characteristics and will shed light on the general category of sensor mix problems.

THEOREM 2.7. Let M be a mixing structure over T and let δ_t be an MSMA decision function consistent with the state of nature. Then the real-valued function whose domain is the set of subsets of S_t and whose value at $V \subset S_t$ is $E[L_t(V, {}_t(V))]$ is monotone, nonincreasing, and has finite minimum value. Further, this minimum value is achieved at $V = S_t$.

Proof: That the set function is monotone and nonincreasing follows easily from Definition 2.6. Since the image of $L_t(S_t,\delta_t(S_t))$ is in Λ, it has finite mean and variance. By examining Definition 2.6, its expected value is easily seen to achieve its minimum value at S_t.

Each subset of S_t may be thought of as contributing a certain amount of assistance to the decision process. If a decision function δ_t is consistent with the state of nature, a convenient indicator of the relative contribution of the various subsets may be defined. For any subset $V \subset S_t$ define

$$G_t(V) = L_t(V,\delta_t(\phi)) - L_t(V,\delta_t(V)).$$

THEOREM 2.8. Let M be a mixing structure over T, and let δ_t be an MSMA decision function consistent with the state of nature. Then the expected value of the random variable $G_t(V)$ is a nonnegative monotone set function which vanishes at ϕ and is bounded above by $E[L_t(\phi,\delta_t(\phi))]$.

Proof: Only the monotonicity will be proved, as the remaining

properties of the expected value of G_t follow immediately from its definition. Let $U,V \subset S_t$ with $U \subset V$. By definition of G_t and the expected value operator

$$E[G_t(U)] = E[L_t(U,\delta_t(\phi))] - E[L_t(U,\delta_t(U))].$$

From Definition 2.6 b)

$$E[G_t(U)] \leq E[L_t(U,\delta_t(\phi))] - E[L_t(V,\delta_t(V))].$$

From Definition 2.6 c)

$$\begin{aligned} E[G_t(U)] &\leq E[L_t(V,\delta_t(\phi))] - E[L_t(V,\delta_t(V))] \\ &= E[L_t(V,\delta_t(\phi)) - L_t(V,\delta_t(V))] \\ &= E[G_t(V)] \quad \text{q.e.d.} \end{aligned}$$

DEFINITION 2.9. Let δ_t be an MSMA decision function consistent with the state of nature. The number

$$100 \frac{E[G_t(V)]}{E[G_t(S_t)]}$$

will be called the <u>relative advantage</u> of $V \subset S_t$ with respect to δ_t.

It is now possible to extend $G_t(\cdot)$ to $\gamma \in \Gamma$. This is accomplished by setting

$$G_t(\gamma) = G_t[g_t(\gamma)]$$

for all $\gamma \in \Gamma$ where $g_t(\gamma) = g_t^*[\{\omega \mid \omega \in \gamma\}]$.

DEFINITION 2.10. $G_t(\gamma)$ will be called the <u>gain of the sensor mix</u> γ. The <u>consistency of the sensor mix</u> γ is defined to be the real number

$$\exp(-\mathrm{Var}[G_t(\gamma)]).$$

It should be noted that the consistency of γ lies in the interval [0,1]. As the variance of the gain approaches zero, the

consistency of the sensor mix approaches its maximum; whereas, as the variance of the gain becomes large, the consistency of γ approaches zero. This agrees with intuitive "feel". It should also be noted that consistency is reminiscent of a probability density.

THEOREM 2.11. The function G_t has the following properties at each $t \in T$:

a) $E[G_t(\phi)] = 0$ where ϕ is the empty sensor mix in Γ.

b) If $\gamma, \mu \in \Gamma$, then $E[G_t(\gamma)] \leq E[G_t(\mu)]$ when $\gamma \subset \mu$.

c) $E[G_t(\gamma)] \geq 0$ for all $\gamma \in \Gamma$.

d) Let $U \subset S_t$ be given. If $\gamma, \nu \in \Gamma$ such that $\gamma, \nu \in (g_t^*)^{-1}(U)$, then $E[G_t(\gamma)] = E[G_t(\nu)]$.

Proof: The proof is elementary and will be omitted.

THEOREM 2.12. Let M be a mixing structure and G_t be the corresponding gain function. Then

a) $\sup\limits_{U \subset S_t} E[G_t(U)]$ is finite and achieved at $U = S_t$.

b) $\sup\limits_{\gamma \in \Gamma} E[G_t(\gamma)]$ exists, is finite, and is achieved for any $\gamma \in (g_t^*)^{-1}(S_t)$.

Proof:

a) $S_t \subset S_t$, hence $G_t(S_t) \in \Lambda$; i.e., it has finite mean and variance. From Theorem 2.7 it follows that $E[G_t(\cdot)]$ is nonnegative and monotone nondecreasing on subsets of S_t. Trivially it follows that

$$\sup_{U \subset S_t} E[G_t(U)] = E[G_t(S_t)].$$

b) Follows from a) and the fact that all sensors

$\beta \in (g^*)^{-1}(S_t)$ have the same expected losses.

Several points should be noted at this time. From Theorem 2.12, it follows that if an MSMA decision function is consistent with the state of nature, one derives the maximum gain from a total knowledge of the state of nature. Secondly, the concept of "gain" allows one to determine relative merits for the various subsets of the state of nature. This is accomplished via the concept of relative advantage contained in Definition 2.9. For each of these subsets, all sensor mixes provide the same expected gain, which is the greatest benefit one can expect to gain from knowledge of this subset of the state of nature. This concept does not allow the determination of that sensor mix which is most favorable for the particular subset of the state of nature variables under consideration; and for this reason it is not yet possible to "balance" gain with "cost" relative to a given sensor mix. More structure is needed.

An MSMA decision function consistent with the state of nature relates the various finite subsets of the state of nature, at a given time, to losses incurred when the values of such subsets are known "a priori" and are utilized in the decision process. Such subsets could represent results from sensor mixes of uniform zero variance. As such, they would represent the idealized "minimum loss" which may be achieved from "a priori" knowledge of the natural state. In general, a sensor mix of such accuracy will not be obtainable or may be prohibitive in cost. Therefore, sensor mixes which are less accurate must be employed.

It may be observed that the concept of "gain" requires little effort to extend it over X. This is due to the fact that each realization of a sensor mix $\gamma \in \Gamma$ corresponds to some $x \in X$.

DEFINITION 2.13. Let M be a mixing structure over T and let δ_t be an MSMA decision function. For any $x \in X$ define

$$G_t(x) = L_t(x, \delta_t(\phi)) - L_t(x, \delta_t(x)).$$

The random variable $G_t(x)$ will be called the gain of x at t when the decision function δ_t is employed. The consistency of $G_t(x)$ is defined to be the real number

$$\exp(-\mathrm{Var}[G_t(x)]).$$

An important relationship between MSMA decision functions and the set Γ of sensor mixes is contained in the following definition.

DEFINITION 2.14.

a) Let U be a subset of the state of nature S_t. An MSMA decision function δ_t is said to link the sensor mixes for U to nature if and only if the following holds. For each sequence of sensor mixes $\{\nu_n \mid n = 1,2,\cdots\}$ in $h^{-1}(U)$ such that

$$\left(\sum_{z_n \in j(\nu_n)} \mathrm{Var}[z_n] \right) \to 0,$$

any sequence $\{x_n \mid n = 1,2,\cdots\}$ of realizations x_n for ν_n has the property that

$$L_t(x_n, \delta_t(x_n)) \to L_t(s_t|_{j(\nu_n)}, \delta_t(s_t|_{j(\nu_n)}))$$

in mean square [14].

b) Let δ_t link the sensor mixes for U to nature. If for all $\nu \in h^{-1}(U)$ there exists a function $w > 0$ in $X_{j(\nu)}$

such that for all $x,y \in X_{j(\nu)}$ satisfying

$$\sum_{q \in j(\nu)} w(q)[y(q) - s_t|_{j(\nu)}(q)]^2$$

$$\leq \sum_{q \in j(\nu)} w(q)[x(q) - s_t|_{j(\nu)}(q)]^2,$$

it follows that

$$E[L_t(y,\delta_t(y))] \leq E[L_t(x,\delta_t(x))],$$

then δ_t is said to <u>strongly link the sensor mixes for U to nature</u>.

It is intuitive that if an MSMA decision function links the sensor mixes for $U \subset S_t$ to nature, then one would expect that a sufficiently small value for the variance of a given sensor mix would in some sense be accompanied by a decrease in the expected loss based on this decision. The weight function w of the strong link to nature reflects that knowledge of the variables comprising the state of nature may not be of equal importance. This is also intuitive.

It is important to note the difference between the concepts of an MSMA decision function <u>consistent with the state of nature</u> and that of an MSMA decision function <u>linking the sensor mixes for U to nature</u>. An MSMA decision function δ_t consistent with the state of nature is characterized by the fact that a decision based on increased knowledge of the state of nature will not result in a decrease of the expected gain, etc. It does not relate to the degree of gain which might be expected as a realization if a given sensor mix approaches the natural state. An MSMA decision function linking the sensor mixes for $U \subset S_t$ to nature reflects this latter behavior.

Both concepts will prove mutually beneficial, and will have considerable importance in the discussions to follow. It will be shown in the next chapter that, in general, these two concepts are not equivalent.

THEOREM 2.15. Let δ_t be an MSMA decision function for the mixing structure M and let $U \subset S_t$. If δ_t strongly links the sensor mixes for U to nature, and $\gamma \varepsilon h^{-1}(U)$, then

$$E[L_t(x,\delta_t(x))] \geq E[L_t(s_t|_{j(\gamma)},\delta_t(s_t|_{j(\gamma)}))]$$

where $x \varepsilon X_{j(\gamma)}$.

Proof: Let x be an arbitrary member of $X_{j(\gamma)}$ and let $y = s_t|_{j(\gamma)}$. The result then follows from Definition 2.14.

It should be noted that the concept of linking (strongly linking) the sensor mixes for $U \subset S_t$ to nature relates to all the sensor mixes for U. However, the concept of an MSMA decision function linking the sensor mixes for U to nature really describes the behavior of the loss function in a neighborhood of the state of nature; i.e., as the associated variance of the sensor mix approaches zero, the loss function converges in mean square to the loss at the state of nature. This all takes place in Λ.

It is now appropriate to demonstrate an important result. If an MSMA decision function links the sensor mixes for $U \subset S_t$ to nature and if ε, η are arbitrary positive constants, then there exists a sensor mix $\gamma \varepsilon h^{-1}(U)$ such that

$$\Pr\{x|\ |L_t(x,\delta_t(x)) - L_t(s_t|_{j(\gamma)},\delta_t(s_t|_{j(\gamma)}))| > \varepsilon\} < \eta.$$

For notational purposes $s = s_t|_{j(\gamma)}$ and $L_t(\cdot) = L_t(\cdot,\delta_t(\cdot))$ will be used from this point. The above probability statement implies

that for a given precision level ϵ, it is possible to choose a sensor mix γ for U whose associated loss differs from the loss due to complete knowledge of the state of nature by as small a value as desired; and this at any predetermined level of probability.

THEOREM 2.16. Let M be a mixing structure for T and $U \subset S_t$. Suppose δ_t is an MSMA decision function linking the sensor mixes for U to nature. Then given any $\epsilon, \eta > 0$ there exists a $\tau > 0$ such that

$$\Pr\{x \mid |L_t(x) - L(s)| > \epsilon\} < \eta$$

whenever $\gamma \varepsilon h^{-1}(U)$ and $\sum_{z \varepsilon \gamma} \mathrm{Var}[z] < \tau$ where $x \varepsilon X_{j(\gamma)}$.

Proof: From Definition 2.14, for any choice of sensor mixes $\{\nu_n \mid n = 1,2,\cdots\}$ in $h^{-1}(U)$ such that

$$\sum_{z_n \varepsilon j(\nu_n)} \mathrm{Var}[z_n] \to 0,$$

then $L_t(x_n) \to L_t(s)$ in mean square. But if a sequence of functions converges in mean square, then it converges in probability. Hence the result follows.

An interesting insight into the mathematical structure presented to this point is afforded by the following theorem which relates the concept of linking the sensor mixes for $U \subset S_t$ to nature and the behavior of the expected loss function.

THEOREM 2.17. Let δ_t be an MSMA decision function for the mixing structure M linking the sensor mixes for $U \subset S_t$ to nature. Let $\gamma \varepsilon h^{-1}(U)$. Suppose that the expected loss function $E[L_t(x)]$ is continuous at $s_t|_{j(\gamma)}$ where $x \varepsilon X_{j(\gamma)}$. Then given any positive constants ϵ, η, there exists a $\tau > 0$ and a sensor mix $\nu \varepsilon h^{-1}(U)$

such that whenever $\sum\limits_{z\varepsilon\nu} \mathrm{Var}[z] < \tau$, then

a) $\Pr\{x \mid |E[L_t(x)] - E[L_t(s)]| \geq \epsilon\} \leq \eta$

b) $E[L_t(x_n)] \to E[L_t(s)]$ in probability where $x_n, n = 1,2,\cdots$ is defined via Definition 2.14.

Proof:

a) The continuity of $E[L_t(x)]$ at $s = s_t\big|_{j(\gamma)}$ gives rise to the existence of a $\tau' > 0$ such that

$$|E[L_t(x)] - E[L_t(s)]| < \epsilon$$

whenever $\|x - s\| < \tau'$. It follows that

$$\{x \mid \|x - s\| < \tau'\} \subset \{x \mid |E[L_t(x)] - E[L_t(s)]| < \epsilon\}$$

or

$$\{x \mid |E[L_t(x)] - E[L_t(s)]| \geq \epsilon\} \subset \{x \mid \|x - s\| \geq \tau'\}.$$

If $\Pr$ is the probability measure determined by the joint distribution of the random vector x on the product measure of $X_{j(\gamma)}$, it is clear that

$$\Pr\{x \mid |E[L_t(x)] - E[L_t(s)]| \geq \epsilon\} \leq \Pr\{x \mid \|x - s\| \geq \tau'\}.$$

By a straightforward extension of Tchebycheff's Inequality [4],

$$\Pr\{x \mid \|x - s\| \geq \tau'\} = \Pr\{x \mid (x - s)^2 \geq (\tau')^2\}$$

$$\leq \frac{\sum\limits_{z\varepsilon\gamma} \mathrm{Var}[z]}{(\tau')^2} .$$

Let τ be such that $\tau < (\tau')^2\eta$. Then any sensor mix $\nu \varepsilon h^{-1}(U)$ such that $\sum\limits_{z\varepsilon\nu} \mathrm{Var}[z] < \tau$ works.

b) This follows immediately from part a) and Definition 2.14.

It should be observed that in the proof of part a) of this theorem, only the continuity of the expected loss function at the state of nature was used.

THEOREM 2.18. Let M be a mixing structure for T and δ_t an MSMA decision function linking the sensor mixes for $U \subset S_t$ to nature. Then for any arbitrary positive constants ϵ, η, there exists a sensor mix $\nu \in h^{-1}(U)$ such that

$$\Pr\{x \mid |G(x) - G(s)| > \epsilon\} < \eta.$$

Proof: The proof follows directly from Definition 2.14 and Theorem 2.16.

Theorem 2.16 effectively states that if an MSMA decision function links the sensor mixes for $U \subset S_t$ to nature, then by choosing a sensor mix $\nu \in h^{-1}(U)$ of sufficient accuracy, the behavior of the loss so determined may be made to approach the behavior of loss when the state of nature is known exactly up to the content of the sensor mix. If, in addition, the decision function is consistent with nature, it is assured that no loss--and possible advantage--will be realized by choosing a more inclusive sensor mix.

It is now appropriate to introduce a "cost" concept into the overall structure. Let M be a mixing structure over T. Let χ be the set of all stochastic processes whose components are in Λ, which are indexed by T, whose components have nonnegative support, and has the property that if $\{\chi_t | t \in T\} \in \chi$, then $\int_T \chi_t dt$ is always defined and is in Λ. Any member of χ will be called a <u>cost process</u> for the mixing structure M. If $T = [t_o, t_e]$ and $t \in T$, then the process

whose component at t is given by

$$\chi_t' = \int_{t_o}^{t} \chi_s d_s$$

will be called the <u>life cycle costing process</u> over T. The set of all life cycle costing processes over T will be denoted by χ'.

DEFINITION 2.19. Let M be a mixing structure. Let χ be the set of all cost processes over T. A map $c_t^*:\Omega \to \chi$ will be called a <u>costing scheme</u> for M. The pair (M,c^*) will be called a <u>cost balanced mixing structure</u>.

DEFINITION 2.20. Let (M,c^*) be a cost balanced mixing structure and let $\gamma \in \Gamma$. The "<u>cost</u>" of the sensor mix γ at t will be defined as

$$c_t(\gamma) = \sum_{z\in\gamma} c_t^*(z).$$

THEOREM 2.21. Let c_t be as in Definition 2.20. Then c_t is a map of Γ into Λ and possesses the following properties:

a) If $\mu,\nu \in \Gamma$ and $\mu \subset \nu$, then $E[c_t(\mu)] \leq E[c_t(\nu)]$.

b) If $\mu,\nu \in \Gamma$ with $\mu \cap \nu = \phi$, then
$E[c_t(\mu \cup \nu)] = E[c_t(\mu)] + E[c_t(\nu)]$.

c) If $\mu,\nu \in \Gamma$ with $\mu \subset \nu$, then
$E[c_t(\mu - \nu)] = E[c_t(\mu)] - E[c_t(\nu)]$.

d) $E[c_t(\cdot)]$ is a nonnegative, nondecreasing, real-valued set function which vanishes at ϕ, and is finitely additive.

Proof: The proof is straightforward and is omitted.

DEFINITION 2.22. Let (M,c^*) be a cost balanced mixing structure, $\gamma \in \Gamma$, and $T = [t_o,t_e]$. The <u>life cycle cost process for the</u>

__sensor mix__ γ will be defined

$$\hat{c}_t(\gamma) \equiv \sum_{z\varepsilon\gamma} \int_{t_o}^{t} c_s(z)ds.$$

THEOREM 2.23. Let $\hat{c}_t$ be as in Definition 2.22. Then

a) $\hat{c}_t$ is a map from $\Gamma \to \Lambda$ and $E[\hat{c}_t(\cdot)]$ is a nonnegative, nondecreasing set function which vanishes at ϕ.

b) If $\mu,\nu \varepsilon \Gamma$ and $\mu \subset \nu$, then

$$E[\hat{c}_t(\mu)] = E[\hat{c}_t(\nu)].$$

c) If $\mu,\nu \varepsilon \Gamma$ and $\mu \cap \nu = \phi$, then

$$E[\hat{c}_t(\mu \cup \nu)] \leq E[\hat{c}_t(\mu)] + E[\hat{c}_t(\nu)].$$

d) If $\mu,\nu \varepsilon \Gamma$ and $\mu \subset \nu$, then

$$E[\hat{c}_t(\nu - \mu)] = E[\hat{c}_t(\nu)] - E[\hat{c}_t(\mu)].$$

Proof: That the image of $\hat{c}_t$ lies in Λ follows immediately from the finiteness of sensor mixes. The remaining portions of the theorem are straightforward and are omitted.

It seems necessary at this point to interrupt the continuity of presentation to clarify a few essential points. In the structures presented so far, one encounters randomness at three different points. There are the stochastic processes in Ω and χ and the random variables in Λ. Each of these probabilistic entities represents random actions; but Λ, χ may in a sense be related to an agency external to the structure considered and has some earmarks of the strategies of a conscious opponent. The processes in χ are the life cycle costing processes for the sensors in Ω or the sensor mixes in Γ, and represent the possibility of increased cost due to accident, effects of hostile environment, sensor failure due to

internal causes, etc. The sensors in Ω are random due to the inability of instruments to observe, record, and transmit information without the possibility of random error.

The MSMA loss function L_t, relating the various estimates of natural states and admissible plans of action with a random "loss" variable at t, contains within its structure much of the cause/effect relationships which constitute a mixing structure. In a real instance this could be very complex. For a given $t \varepsilon T$, the MSMA loss function at t contains an entire rationale of operation for the system, as well as the sensitivity of the system to fluctuations in the physical observables. This loss function depends only on the action taken and the prevailing state of nature, and not on the observed state of nature as indicated by $x \varepsilon X$. The sole link between an observation "x" and the "loss" experienced is through the MSMA decision function. It is very possible that such a function produces a minimal loss based on an entirely biased knowledge of the state of nature.

The sensitivity of "loss" due to variations in the state of nature, or to observational errors in the pertinent states of nature may arise from inherent characteristics of the loss function itself or from interactions between the loss function and an MSMA decision function. However, since any MSMA decision function maps X into $\hat{A}_t$ which has neither measure nor topological characteristics, only the sensitivity as manifested through the behavior of L_t can be studied.

It is now worthwhile to reexamine the concept of gain as related

to the idea of sensor mixes.

THEOREM 2.24. Let M be a mixing structure and δ_t be an MSMA decision function. Let G_t be the gain for $x \varepsilon X$ as defined in Definition 2.13. If δ_t strongly links the sensor mixes for $U \subset S_t$ to nature, then

a) For any $x \varepsilon X_{j(\gamma)}$, $E[G_t(x)] \leq E[G_t(s)]$ where $\gamma \varepsilon h^{-1}(U)$.

b) If $x,y \varepsilon X_{j(\gamma)}$ and $0 < w \varepsilon X_{j(\gamma)}$ is such that

$$\sum_{q\varepsilon j(\gamma)} w(q)[y(q) - s(q)]^2 \leq \sum_{q\varepsilon j(\gamma)} w(q)[x(q) - s(q)]^2, \text{ then}$$

$E[G_t(y)] \leq E[G_t(x)]$.

c) Let $<x_n>$ be any sequence of members of $X_{j(\gamma)}$. If $n > m$ implies

$$\|x_n - s\| \leq \|x_m - s\|,$$

then the sequence $<E[G_t(x_n)]>$ is a monotone increasing sequence of real numbers converging in mean square to $E[G_t(s)]$.

Proof:

a) It follows trivially from Theorem 2.15 a).

b) and c) follow from Definition 2.14.

DEFINITION 2.25. Let (M,c^*) be a cost balanced mixing structure over T and let δ_t be an MSMA decision function. For $\gamma \varepsilon \Gamma$, the random variable

$$F_t(\gamma) = G_t(\gamma) - kc_t(\gamma)$$

will be called the <u>instantaneous figure of merit</u> for γ at t. If $E[F_t(\gamma)] > 0$, γ will be called an <u>acceptable sensor mix</u> at t for the MSMA decision function δ_t. $\gamma \varepsilon \Gamma$ is said to have <u>greater</u>

__relative merit__ than $\mu \in \Gamma$ at t if and only if

$E[F_t(\gamma)] > E[F_t(\mu)]$.

It should be noted that in the above definition it is assumed that G_t and kc_t are expressed in the same value units, whatever these may be. k is the constant conversion factor required to accomplish this.

THEOREM 2.26. Let (M,c^*), δ_t, and F_t be as in Definition 2.25. Then if δ_t is consistent with the state of nature, $F_t:\Gamma \to \Lambda$ is such that $F_t(\phi) = 0$ and for $\mu \in (g_t^*)^{-1}(S_t)$ where g_t^* is defined in Definition 2.10 we have for $\gamma \in \Gamma$

$$E[G_t(\gamma)] \leq E[G_t(\mu)].$$

Proof: $F_t(\phi) = 0$ follows immediately from Definition 2.25. $E[c_t(\gamma)] \geq 0$ by Theorem 2.21 d) for all $\gamma \in \Gamma$. By Theorem 2.12 for all $\gamma \in \Gamma$ and all $\mu \in (g_t^*)^{-1}(S_t)$

$$E[G_t(\gamma)] \leq E[G_t(\mu)].$$

Hence the result follows.

Recall from Theorem 2.11 that $E[G_t(\gamma)] = E[G_t(\mu)]$ where $\gamma,\mu \in (g_t^*)^{-1}(U)$. Hence to obtain the maximum expected instantaneous figure of merit for a subset $U \subset S_t$, it is necessary to choose a sensor mix $\gamma \in \Gamma$ which provides the necessary information about $U \subset S_t$ with the minimal expected cost. This may not be the most feasible course of action, however, since the gain function may have a large variance. Thus it is necessary to form a tradeoff between the consistency of the gain function and the cost function. Toward this end the presentation continues.

Observe that in a straightforward manner the definition of

figure of merit may be extended to $x \in X$.

DEFINITION 2.27. Let (M,c^*) be a cost balanced mixing structure over T and δ_t be an MSMA decision function consistent with the state of nature. For $\gamma \in \Gamma$, and $x \in X_{j(\gamma)}$ define

$$F_t(x) = G_t(x) - kc_t(\gamma).$$

As before, k is a constant conversion factor. Then $F_t(x)$ will be called the <u>figure</u> <u>of</u> <u>merit</u> for γ at t. The real number $E[F_t(x)]$ will be called the <u>expected</u> <u>figure</u> of merit of γ at t.

Recall from Theorem 2.18 that if given an $\varepsilon,\eta > 0$ and an MSMA decision function δ_t linking the sensor mixes for $U \subset S_t$ to nature, it is possible to obtain a sensor mix $\nu \in h^{-1}(U)$ such that

$$\Pr\{x \mid |G(x) - G(s)| > \varepsilon\} < \eta$$

where η depends on the variance of the sensor mix ν. As the variances related to a sensor mix approach zero, the gain function approaches that obtained from complete information concerning the state of nature. Thus if an MSMA decision function is both consistent with the state of nature and strongly links the sensor mixes for $U \subset S_t$ to nature, then as $x_n \to s$, $E[G(x_n)]$ converges in mean square to $E[G(s)]$ monotonically. In this case it may not prove cost effective to obtain sensors more accurate than a certain value; and balance between cost of the sensors and the consistency of the gain may be possible.

To this point the effort has been largely directed toward the formalization of a mixing structure and its behavior, at some fixed t in T; that is, the instantaneous behavior. Concern will now be with the cumulative behavior at each t in T. If it is desired to

determine the advantage of employing a sensor mix γ up to some point $t \varepsilon T$, it should follow that the accumulated gain must be balanced against the cost to t of the operation. The following definition specifies the sort of mixing structures for which such an action is possible.

DEFINITION 2.28. Let M be a mixing structure over T and let $\{\delta_t \mid t \varepsilon T\}$ be a decision scheme for M. The mixing structure M will be said to have a <u>summable</u> <u>progression</u> <u>of</u> <u>losses</u> with respect to $\{\delta_t \mid t \varepsilon T\}$ if and only if for any $\gamma \varepsilon \Gamma$, the process

$$\{L_t(x_t,\delta_t(x_t)) \mid t \varepsilon T\}$$

is Lebesque-Stieltjes integrable. In this case the scheme $\{\delta_t \mid t \varepsilon T\}$ will be said to <u>determine</u> <u>a</u> <u>summable</u> <u>progression</u> <u>of</u> <u>losses</u>.

Note that if $\{\delta_t \mid t \varepsilon T\}$ determines a summable progression of losses, one may define new functions L_t^*, G_t^*, and F_t^* by allowing the values of these functions at $t \varepsilon T$ to be the appropriate integrals to $t \varepsilon T$ of L_t, G_t, and F_t. The existence of the functions G_t^* and F_t^* follows from the fact that L_t^* is a summable progression of losses and the fact that the cost function is monotonically nondecreasing.

Concern will now center on fixed sensor mixes over T.

DEFINITION 2.29. The function F_t^* will be called the <u>cumulative</u> <u>figure</u> <u>of</u> <u>merit</u> for the sensor mix γ. The real number $E[F_t^*(\gamma)]$ will be called the <u>expected</u> <u>cumulative</u> <u>figure</u> <u>of</u> <u>merit</u>.

THEOREM 2.30. The cumulative figure of merit F^* for a sensor mix γ is a real-valued set function on Γ such that $F^*(\phi) = 0$.

Proof: The proof is elementary and is omitted.

Several observations may be made at this time. The cumulative figure of merit may not be a nondecreasing function of time. This is easily seen by examining the appropriate definition. A sensor mix γ at a fixed $t \varepsilon T$ may have a larger expected cumulative figure of merit than a sensor mix $\mu \varepsilon \Gamma$, but may alter drastically at some other $t \varepsilon T$.

The expected cumulative figure of merit may be large, but the sensor mix may not be desirable due to large fluctuations or variance of the cumulative gain function.

It is at this point that cost and gain can be "balanced". By examining the totality of acceptable sensor mixes for M which is related to the expected cumulative figure of merit, one proceeds as follows.

Suppose the most important consideration for choosing an acceptable sensor mix is the consistency of the gain. Then by choosing an acceptable variance for the gain function G^*, one determines a sensor mix of minimal cost which gives this result.

However, if the life cycle cost is the essential item under consideration, one may choose a sensor mix that yields the largest expected cumulative gain or the one with the most consistency of gain.

Thirdly, it may be necessary to have both a limit on the life cycle cost aspect and to consider the consistency of gain. It is now possible to graph life cycle cost versus the cumulative consistency for gain and thus consider possible tradeoffs.

Sufficient structure has been developed at this time to examine the Problems I, II, and III as defined earlier. However, the method used to solve these problem areas depends on the particular application of interest. Certain applied problems using this theory are currently undergoing examination and investigation.

CHAPTER III

AN ILLUSTRATIVE EXAMPLE

To this point the existence of a mixing structure and related entities has not been demonstrated. The purpose of this section is to provide a nontrivial example which will illustrate the existence of such and will provide a valuable amount of structural insight.

In the following T will be taken as a nontrivial finite closed interval of real numbers, and P will be a set of finite cardinality n, where n is greater than one. Now, let $\{f_p \mid p \in P\}$ be an indexed set of distinct real-valued functions defined and bounded on T. The set $\{f_p(t) \mid p \in P\}$ will be taken as the state of nature at t, so that the set $\{f_p(t) \mid t \in T\}$ becomes the evolution of the natural state f_p over T.

Let (R,L,m) be the Lebesque real measure space, and let Ω and Λ be defined as in Chapter II. Recall that Ω is the set of all stochastic processes on (R,L,m) with the property that if $\{\omega_t \mid t \in T\}$ is in Ω, then there exists a natural state f_p whose evolution is $\{E(\omega_t) \mid t \in T\}$. Also recall that Λ is the set of all random variables on (R,L,m) of finite mean and variance.

The structures developed via each of the following theorems or observations will be assumed in all subsequent constructions.

OBSERVATION 3.1. Let A_t be the set of real numbers and $\hat{A}_t$ be all subsets of A_t of cardinality less than or equal to that of P. Then $\{(A_t,\hat{A}_t) \mid t \in T\}$ may be taken as an action family over T, where $\hat{A}_t$ represents admissible plans of action for each $t \in T$.

OBSERVATION 3.2. Let Ω, Λ, and $\{(A_t,\hat{A}_t) \mid t \in T\}$ be as previously defined. Let X be defined in context; i.e., $X = \bigcup_{Q \subset P} X_Q$ where $X_Q = \{x \mid x : Q \to R\}$. Consider the function $d_t : X \times \hat{A}_t \to R$ by

$$d_t(x,B_t) = \begin{cases} M(t) & x \in X,\ B_t = \phi \\ M(t)\left[1 - \exp\left\{- \sum_{p \in P} \prod_{b \in B_t} \alpha_p(t)[b - f_p(t)]^2\right\}\right] & \text{otherwise.} \end{cases}$$

Here $M(t)$ is a positive real number and $\{\alpha_p(t) \mid p \in P\}$ is the value at t of a set of positive real-valued functions $\{\alpha_p \mid p \in P\}$ defined on T. The vector $\{\alpha_p \mid p \in P\}$ is a weight function whose role is to assign a relative importance to the various states comprising the state of nature.

Obviously d_t is well-defined and nonnegative for all $t \in T$. Define a map for each $t \in T$, $L_t : X \times \hat{A}_t \to \Lambda$ by

$$L_t(x,B_t) = \begin{cases} N(d_t(x,\phi),\sigma^2) & B_t = \phi \\ N(d_t(x,B_t),\sigma^2) & \text{otherwise} \end{cases}$$

where $N(K,\sigma^2)$ is a normal random variable with mean K and variance σ^2.

Clearly the image of L_t is in Λ so that $\{L_t \mid t \in T\}$ defines a progression of losses for the cited structure. Hence L_t is an MSMA loss function. It might also be noted that $B_t = \{f_p(t) \mid p \in P\}$ is itself an admissible plan of action at t.

THEOREM 3.3. The function d_t as defined in the previous observation is characterized by

a) The image of d_t lies in $[0,M(t)]$ for all $t \in T$ and is equal to $M(t)$ at $B_t = \phi$.

b) $d_t(x,B_t)$ has a minimum of 0 when action B_t equals the state of nature $\{f_p(t) \mid p \in P\}$.

Proof:

a) Part a) is by definition.

b) When $B_t = \{f_p(t) \mid p \in P\}$, each b in B_t is equal to $f_p(t)$ for some $p \in P$. Hence

$$\sum_{p\in P} \prod_{b\in B_t} \alpha_p(t)[b - f_p(t)]^2 = \sum_{p\in P} 0 = 0.$$

It follows that $d_t(x,B_t)$ is also zero.

COROLLARY 3.4. The expected value of the random variable L_t possesses the following properties:

a) The image of $E[L_t(\cdot,\cdot)]$ lies in $[0,M(t)]$ for all t in T.

b) $E[L_t(x,B_t)] = 0$ when $B_t = \{f_p(t) \mid p \in P\}$.

Proof: This is an immediate consequence of Theorem 3.3 and the definition of $L_t(\cdot,\cdot)$.

THEOREM 3.5. The sextuple

$$[\{f_p \mid p \in P\}, \{(A_t,\hat{A}_t) \mid t \in T\}, \Omega, X, \{L_t \mid t \in T\}, \Lambda]$$

forms a nontrivial mixing structure.

Proof: The proof follows directly from Observations 3.1 and 3.2.

THEOREM 3.6. Any map $\beta : R \to R$ determines a set of MSMA decision functions for the mixing structure M in the previous theorem.

Proof: Let $\beta : R \to R$ be an arbitrary function. Let $\delta_t(\phi) = \phi$. For each $x \varepsilon X$ there exists a subset $Q \subset P$ such that $x \varepsilon X_Q$. Define

$$\delta_t(x) = \{\beta[x(q)] \mid q \varepsilon Q \subset P\}.$$

Certainly $\delta_t(x)$ defines a subset of A_t, and, in fact, of $\hat{A}_t$. Thus $\delta_t(x)$ is an MSMA decision function. Since β was chosen arbitrarily, the result follows.

COROLLARY 3.7. Let J, K be any two natural numbers. Then any transformation β from Euclidean J-space E^J to Euclidean K-space E^K determines a set of MSMA decision functions for the mixing structure M.

Proof: Let $\beta : E^J \to E^K$ be an arbitrary transformation. Let $\delta_t(\phi) = 0$ and $x \varepsilon X$. Thus there exists a subset $Q \subset P$ such that $x \varepsilon X_Q$. Define

$$\delta_t(x) = \{r_1(\beta[(x(q),0,\cdots 0)]) \mid q \varepsilon Q \subset P, (x(q),0,\cdots 0) \varepsilon E^J\}$$

where r_1 is the projection from E^K onto its first coordinate. The fact that $\delta_t(x)$ is an MSMA decision function follows from Theorem 3.6.

It should be noted that each transformation β from E^J to E^K determines many MSMA decisions for the cited mixing structure. For example, instead of projecting E^K onto its first coordinate, one may use any other coordinate projection.

THEOREM 3.8. For each subset $U \subset S_t$, where S_t is the state of nature at t, define $\delta_t(U) = U$. Then δ_t is an MSMA decision function consistent with the state of nature at t.

Proof: Clearly $\delta_t(U)$ is an MSMA decision function for each $U \subset S_t$. To show that this decision function is consistent with the state of nature, it suffices to examine the function d_t as defined in Observation 3.2.

Case I: If $U = \phi$, then $\delta_t(U) = \phi$ and $d_t(x,\phi) = M(t)$. And all three parts of Definition 2.6 are clearly satisfied.

Case II: If $U \neq \phi$, then for each $p \in P$ such that $f_p(t) \in U$, $b = f_p(t)$. Hence

$$\prod_{b\in\delta_t(U)} \alpha_p(t)[b - f_p(t)]^2 = 0.$$

Let $U \subset V$, then $f_p(t) \in U \subset V$. For each $p \in P$ such that $f_p(t) \in V - U$,

$$\prod_{b\in\delta_t(U)} \alpha_p(t)[b - f_p(t)]^2 \neq 0.$$

By letting $Q_1 = \{p \in P \mid f_p(t) \notin V\}$ and $Q_2 = \{p \in P \mid f_p(t) \notin U\}$, it follows that

$$\begin{aligned}
E[L_t(V,\delta_t(V))] &= d_t(V,\delta_t(V)) \\
&= M(t)[1 - \exp\{-\sum_{p\in P} \prod_{b\in\delta_t(V)} \alpha_p(t)[b - f_p(t)]^2\}] \\
&= M(t)[1 - \exp\{-\sum_{p\in Q_1} \prod_{b\in\delta_t(V)} \alpha_p(t)[b - f_p(t)]^2\}] \\
&< M(t)[1 - \exp\{-\sum_{p\in Q_2} \prod_{b\in\delta_t(U)} \alpha_p(t)[b - f_p(t)]^2\}] \\
&= d_t(U,\delta_t(U)) = E[L_t(U,\delta_t(U))].
\end{aligned}$$

Thus part a) of Definition 2.6 is satisfied. Part b) follows from the fact that $d_t(U,\delta_t(U)) = d_t(V,\delta_t(U))$ and the proof of part a) above. Part c) is trivial. Thus $\delta_t(\cdot)$ is an MSMA decision function consistent with the state of nature.

Note: Let f_p be a natural state. Recall that the set of sensors for f_p is $\{\omega \in \Omega \mid \omega \in (h^*)^{-1}(f_p)\}$ where h^* represents the following relationship. Let $\omega = \{\omega_t \mid t \in T\}$. Then $h^*(\omega) = f_p$ where f_p is the state variable whose evolution is $\{E(\omega_t) \mid t \in T\}$.

OBSERVATION 3.9. Let $\gamma \in \Gamma$ be a sensor mix and $x \in X_{j(\gamma)}$. Define $\delta_t(x) = x$. Observe that $\delta_t(x)$ is an MSMA decision function for M, and if all the sensors in γ have zero variance, then $\delta_t(x) = \delta_t(U)$ for an appropriate $U \subset S_t$ where $\delta_t(U)$ is defined via Theorem 3.8.

THEOREM 3.10. The MSMA decision function $\delta_t(\cdot)$ defined in Observation 3.9 links the sensor mixes for $U \subset S_t$ to nature.

Proof: For $\phi = U \subset S_t$, the theorem is vacuously true. Let $\phi \neq U \subset S_t$ and $\{\nu_n \mid n = 1,2,\cdots\}$ a sequence of sensor mixes in $h^{-1}(U)$ such that $\sum\limits_{z_n \varepsilon \nu_n} \mathrm{Var}[z_n] \to 0$. Let $\{x_n \mid n = 1,2,\cdots\}$ be any sequence of realizations x_n for ν_n. From Theorem 3.3, $d_t(s,\delta_t(s)) = 0$. Hence

$$R(x_n) = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \left\{\exp\left[-\frac{(y-d_t(x_n,\delta_t(x_n))^2}{2\sigma^2}\right] - \exp\left[-\frac{(y-d_t(s,\delta_t(s))^2}{2\sigma^2}\right]\right\}^2 dy$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \left\{\exp\left[-\frac{(y-d_t(x_n,\delta_t(x_n))^2}{2\sigma^2}\right] - \exp\left[-\frac{y^2}{2\sigma^2}\right]\right\}^2 dy,$$

which is integrable since the square of an integrable function is integrable. Thus $R(x_n)$ is a continuous function of x_n. Since

$$\lim_{n\to\infty} d_t(x_n,\delta_t(x_n)) = 0,$$

$\lim_{n\to\infty} R(x_n) = 0$, i.e., $L_t(x_n)$ converges to $L_t(s)$ in mean square.

Thus δ_t is an MSMA decision function linking the sensor mixes for U to nature. This holds for all subsets $U \subset S_t$ and thus the theorem follows.

OBSERVATION 3.11. Let $B = \{f_p(t) \mid p \varepsilon P\}$. This is an optimal plan of action for M in the sense that it results in a unique absolute minimal value for the expected value of the MSMA loss function $L_t(\cdot,\cdot)$.

Proof: $E[L_t(x,B)] = d_t(x,B)$. Clearly $d_t(x,B)$ is bounded below by zero. Also for $B = \{f_p(t) \mid p \varepsilon P\}$, $d_t(x,B) = 0$. Thus only the uniqueness of B needs verification. $d_t(x,B) = 0$ implies that

$$\exp\left\{-\sum_{p\varepsilon P} \prod_{b\varepsilon B} \alpha_p(t)[b - f_p(t)]^2\right\} = 1.$$

But $\exp(-y) = 1$ implies that $y = 0$. Hence

$$\sum_{p\varepsilon P} \prod_{b\varepsilon B} \alpha_p(t)[b - f_p(t)]^2 = 0.$$

Since $\alpha_p(t) > 0$ and $[b - f_p(t)]^2 \geq 0$, each term of the summation must be zero. This in turn implies $[b - f_p(t)] = 0$ for each $b \varepsilon B$. This can only occur if $b = f_p(t)$ for some $p \varepsilon P$. Since we are summing over P, this must be true for all $p \varepsilon P$. Therefore, $B = \{f_p(t) \mid p \varepsilon P\}$ is unique.

To show that the concepts of an MSMA decision function consistent with the state of nature and of an MSMA decision function linking the sensor mixes for $U \subseteq S_t$ to nature are not equivalent, a

slight modification of the MSMA loss function proves satisfactory.

OBSERVATION 3.12. Consider

$$d_t(x,B) = \begin{cases} M(t) + 1 & B = \phi \\ -1 & x = s \\ M(t)\left[1 - \exp\left\{-\sum_{p\varepsilon P} \prod_{b\varepsilon B} \alpha_p(t)[b - f_p(t)]^2\right\}\right] & \text{otherwise} \end{cases}$$

where $M(t)$ and $\alpha_p(t)$ are defined in Observation 3.2. Let $L_t(x,B)$ and $\delta_t(x)$ be as previously defined in Observations 3.2 and 3.9 respectively. To show $\delta_t(\cdot)$ is an MSMA decision function consistent with the state of nature, it suffices to use a proof identical to that of Theorem 3.8. Since $d_t(\cdot,\cdot)$ has a jump discontinuity at the state of nature,

$$\lim_{n\to\infty} \| d_t(x_n,\delta_t(x_n)) - d_t(s,\delta_t(s)) \| > 0$$

for any sequence $\{x_n \mid n = 1,2,\cdots\}$ in X which converges to s. Thus

$$\lim_{n\to\infty} \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \left\{\exp\left[-\frac{(y-d_t(x_n,\delta_t(x_n)))^2}{2\sigma^2}\right] - \exp\left[-\frac{(y-d_t(s,\delta_t(s)))^2}{2\sigma^2}\right]\right\}^2 dy > 0$$

and $\delta_t(\cdot)$ does not link (strongly link) the sensor mixes for $U \subset S_t$ to nature for any nonempty $U \subset S_t$.

OBSERVATION 3.13. Consider

$$d_t(x,B) = \begin{cases} M(t) & B = \phi \\ M(t)\left[1 + \exp\left\{-\sum_{p\varepsilon P} \prod_{b\varepsilon B} \alpha_p(t)[b - f_p(t)]^2\right\}\right] & \text{otherwise} \end{cases}$$

where $M(t)$ and $\alpha_p(t)$ are as previously defined. Let $L_t(x,B)$ and $\delta_t(x)$ be as previously defined in Observations 3.2 and 3.9 respectively. By mimicking the proof to Theorem 3.10, it is easy to demonstrate that $\delta_t(\cdot)$ is an MSMA decision function linking the sensor mixes for each nonempty subset $U \subset S_t$ to nature. That $\delta_t(\cdot)$ is an MSMA decision function not consistent with the state of nature follows by noting if $U \subset V$, $U,V \subset S_t$, then

$$E[L_t(U,\delta_t(U))] > E[L_t(V,\delta_t(V))].$$

This contradicts part b) of Definition 2.6.

OBSERVATION 3.14. Let $d_t(x,B)$, $L_t(x,B)$, and $\delta_t(x)$ be as defined in Observation 3.13. Then $\delta_t(\cdot)$ is an MSMA decision function which does <u>not</u> strongly link the sensor mixes for $U \subseteq S_t$ to nature for any nonempty $U \subset S_t$. This follows by noting that for all nonempty sensor mixes $\gamma \varepsilon \Gamma$

$$\begin{aligned} d_t(x,\delta_t(x)) &= E[L_t(x,\delta_t(x))] > E[L_t(s,\delta_t(s))] \\ &= d_t(s,\delta_t(s)) \end{aligned}$$

for $x \neq s$, $x \varepsilon X_{j(\gamma)}$. This contradicts part b) of Theorem 2.15. Thus the example contained in Observation 3.12 demonstrates that the concept of linking γ to nature does not imply the concept of strongly linking to nature.

This section thus provides a nontrivial example of a mixing structure and related entities. It also demonstrates the fact that the concepts of an MSMA decision function being consistent with the state of nature and linking the sensor mixes for $U \subset S_t$ to nature are not equivalent.

To highlight this example and some of the structure developed

herein, a simplified and somewhat naive version of an artillery problem may be used. This problem is concerned with the ability to hit a certain target with an artillery projectile. It is assumed that the only contribution to the "missing" of the target is due to a lack of knowledge concerning the meteorological parameters of atmospheric temperature, wind speed, wind direction, and atmospheric pressure. Each of these parameters is further assumed to be characterized by a single real number, i.e., the "ballistic number". The state of nature S_t then consists of these four meteorological parameters.

Since the artillery projectile is unguided, the launcher setting required to hit the target must include compensation for the four meteorological parameters of interest. To accomplish this, the artillery officer in charge must decide what values should be used for these parameters. This will constitute his action set. These values are then utilized as input to some undefined black box which converts them into launcher settings. The projectile is then fired on these settings.

For this problem, the set of actions at t can be taken as the real line and the admissible actions at t the subsets of the real line of cardinality less than or equal to four. This coincides with the family of actions as defined in Observation 3.1.

To obtain estimates about the state of nature, various meteorological sensors are used. These range from the very simple, such as a thermometer, to the very sophisticated, such as radars and satellites. Each of these sensors has its own probabilistic

characteristics. Let Ω, Λ, and $\{(A_t, \hat{A}_t) \mid t \in T\}$ be as previously defined and let X be defined in context.

Let the loss function at t, L_t, be an assessment of the "miss distance" for the projectile. It is intuitive that the four meteorological parameters contribute unequally to the miss distance, and that the expected loss due to lack of knowledge of the state of nature should be maximal. The loss function as defined in Observation 3.2 has these properties. As increased information concerning the state of nature is obtained, the expected loss should approach zero, which would be the case when complete knowledge of the state of nature is known. Thus this loss function is applicable to the stated problem.

The sextuple $[\{f_p \mid p \in P\}, \{(A_t, \hat{A}_t) \mid t \in T\}, \Omega, X, \{L_t \mid t \in T\}, \Lambda]$ thus forms a mixing structure as indicated in Theorem 3.5.

When the admissible action is equal to the state of nature, the expected loss is zero; and if the chosen action is the standard plan of action, the expected loss is maximal $M(t)$.

Let Γ be the set of all sensor mixes, $\gamma \in \Gamma$, and $x \in X_{j(\gamma)}$. By letting $\delta_t(x) = x$, one obtains an MSMA decision function for this problem. Since this is a mixing structure without lag, it is easy to verify that $\delta_t(\cdot)$ is an MSMA decision function which is both consistent with the state of nature and links every sensor mix for $U \subset S_t$ to nature. In fact, if the admissible action corresponds to a sensor mix with zero variance, a unique absolute minimal value for the MSMA loss function is obtained as in Observation 3.11.

Thus it may be seen that this simplified version of the artillery problem demonstrates the applicability of both this illustrative example and the theory as developed in Chapter II. In fact, it is not difficult to visualize a wide variety of other practical applications.

CHAPTER IV

OPEN PROBLEMS

In this chapter a discussion of selected open problems is presented.

It should be observed that at no time in the mathematical development presented previously was there any discussion of methods of derivation for a decision function yielding a feasible course of action. This was in the large intentional, since the choice of a decision function may well depend on the particular application of interest; i.e., on the form of the MSMA loss function itself. Hence a family of open problems arise in the selection of decision functions leading to those feasible courses of action particular to a given loss function. Possible approaches involve the use of dynamic programming or sequential decision theory. (See [1] or [15].)

The relation of this structure and other known theories may be more comprehensively investigated with similarities and differences precisely delineated. Pertinent theories include optimal stopping [5], sampling theory [2], various economic theories such as organization theory [12], team decision theory [16], sequential decision theory [15], differential games [6] and [8-10], information theory [18], etc.

In this presentation, only mixing structures without lag are considered. This implies an instantaneous occurrence for the following: observations made by sensors for the subset of the state of nature, a choice of an MSMA decision function based on these observations, and leading to the initialization of an admissible

plan of action, and the realization of the loss function based on this plan of action.

In physical situations, this assumption is often unrealistic as there usually exists some delay between these occurrences. This leads to certain questions as yet unanswered.

In the following, a mixing structure with lag will be introduced, and certain problem areas associated with this concept will be identified.

Let observations concerning the state of nature be taken at time t; but the plan of action initiated at time $t + \tau$ be based on this "lagged" information. This may result in an increase of instantaneous loss at $t + \tau$. Suppose now that an MSMA decision function δ_t is chosen at time t, but a plan of action cannot be initiated until time $t + \tau$, which results in an instantaneous loss $L_{t+\tau}(x, \delta_t(x))$ at $t + \tau$. It may be seen that this loss function produces a result identical to the loss associated with a different MSMA decision function chosen at time $t + \tau$, and an instantaneous plan of action initiated at that time. In both cases the MSMA decision function may be considered to have a lag of τ.

It can be seen that the set of decision functions leading to a feasible or an optimal plan of action may vary from those which now arise in the problems considered in the main text. Marschak and Radner [12] have studied such problems for special cases relating to organizational theory. General solutions to this problem do not exist at present.

Other problem areas for the mixing structure with and without

lag are as follows. Suppose that values of the MSMA loss function at $t + \tau$ are correlated to those at t. One immediate prc is the choice of a decision scheme which takes this behavior into account. Suppose further that the plan of action at t may be influenced by prior observations and/or prior losses. By placing appropriate assumptions on such relationships, the three general problem areas addressed in the text are altered in context to new problems. Many applications require such considerations.

In military applications, one must face a conscious opponent. This opponent may also be sensing information concerning the state of nature. A natural question arises as to how this affects appropriate strategy. Thus for varying types of military applications, the structure is changeable, and new problems arise.

The partial inclusion of the attributes derivable from a conscious opponent should be reflected in the loss function. Knowledge of the natural state may give rise to advantage. The opponent in turn may attempt to counteract this advantage. Both the loss and cost functions could be altered by such an attempt. Such actions will be reflected in the progression of losses, and should be investigated.

The merit of cost and operational effectiveness studies in the design of military systems is well established, but often involves intangibles, and complicated measures of effectiveness. For non-military applications both cost and effectiveness are expressed in a monetary standard. Although the structures here presented were designed for military applications, any physical system in which

actions are based on analyses of observations concerning some state of nature may be addressed in a similar fashion.

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